Int. J. Heat Mass Transfer. Vol. 27, No. 1, pp. 29–37, 1984
Printed in Great Britain

0017-9310/84 $3.00 + 0.00

THE EFFECTS OF HEAT GENERATION AND WALL INTERACTION ON FREEZING AND MELTING IN A FINITE SLAB

F. B. CHEUNG, T. C. CHAWLA and D. R. PEDERSEN
Reactor Analysis and Safety Division, Argonne National Laboratory, 9700 South Cass Avenue, Argonne, IL 60439, U.S.A.

(*Received* 4 *February* 1983 *and in revised form* 18 *April* 1983)

Abstract—The processes of freezing and melting occurring in a heat-generating slab bounded by two semi-infinite cold walls is studied numerically. The method of collocation is employed to solve the various sets of governing equations describing the unsteady behavior of the system during different periods of time. Depending on the rate of internal heat generation and the thermal properties of the wall and the slab, several changes may take place in the system. These changes, as indicated by the transient locations of the solid–liquid interface, include transitions from freezing directly to melting, from freezing to cooling without phase change, from cooling to heating without phase change, and from heating to melting. Numerical predictions of the occurrence of these transitions, the rates of freezing and melting, and the duration of the transients are obtained as functions of several controlling dimensionless parameters of the system. Comparison is made with the case of a heat-generating sphere to further explore the effect of system geometry.

NOMENCLATURE

A dimensionless heat generation rate, equation (15a)
Cp_s specific heat of the crust
ΔH_f heat of fusion
k_s thermal conductivity of the crust
k_w thermal conductivity of the wall
L half width of the slab
q rate of volumetric heating
Ste Stefan number, equation (15c)
t time
T_0 initial temperature of the liquid
T_∞ initial temperature of the wall
T_l liquid temperature
T_m melting point of the crust
T_s solid crust temperature
T_w wall temperature
x physical coordinate.

Greek symbols
α_s thermal diffusivity of the crust
α_w thermal diffusivity of the wall
ρ_s crust density
δ crust thickness
Δ dimensionless crust thickness, equation (8)
Δ_m maximum crust thickness
β stretching parameter for numerical computation
η dimensionless coordinate, equation (10)
τ dimensionless time, equation (8)
ϕ dimensionless liquid temperature, equation (9)
ϕ_0 liquid superheat factor, equation (15b)
ψ dimensionless crust temperature, equation (9)
θ dimensionless wall temperature, equation (9).

Subscripts
l liquid region
m melting condition or otherwise specified
0 initial condition
s crust region
w wall region.

1. INTRODUCTION

HEAT transfer problems involving a phase change due to solidification or melting are important in many industrial applications such as in the casting of metals, the making of ice, the freezing and thawing of meat blocks, the drilling of high ice-content soil, the design of cryogenic heat exchangers, the storage of thermal energy, and the safety studies of nuclear reactors. Since Stefan's pioneering work on the growth of sea ice [1], many solutions of heat diffusion-controlled freezing or melting problems have been reported as can be found in refs. [2–4]. Useful analytical and numerical techniques for treating such problems have been recently reviewed by Wilson *et al.* [5], Özişik [6], and Lunardini [7].

One common feature of phase change problems is that the location of the solid–liquid interface is not known *a priori* and must be determined during the course of analysis. Mathematically, the interface motion is expressed implicitly in an equation for the conservation of thermal energy at the interface. This introduces a non-linear character to the system which renders each problem somewhat unique. The exact solution of phase change problems is limited almost exclusively to the case in which the heat transfer regions are semi-infinite in extent. For a finite slab, analytical solutions exist only for the situation with particularly simple initial and boundary conditions [8–10]. More generally, exact or similarity solutions are not available so that approximate or numerical methods must be used to solve the problem.

This paper is a numerical study of the transient behavior of a heat-generating substance that is initially molten and trapped between two semi-infinite walls at temperatures below the freezing point of the substance.

The motivation of this work is to simulate the situation of freezing and melting of disrupted core materials with decay heating in the gaps of cold MgO bricks within the ex-vessel cavity or in the thick-walled channel region surrounding a liquid-metal fast reactor core. Should there be no internal heat generation due to decay heating, solidification of the molten core debris would proceed such that the entire region would become frozen and remain so afterward. With decay heating, however, the frozen core materials could not remain solid indefinitely. As the thermal boundary layer in the wall becomes thick enough, transient conduction through the wall is not sufficient to remove all the heat generated in the core debris region. As a result, melting of the frozen materials will occur.* The rate of melting actually increases in time since the conductive wall heat flux is diminishing. The entire core debris region will become molten again when melting is complete. Depending on the level of decay heating and the wall effect, melting of the frozen core materials may occur before or after the entire region freezes solid. In nuclear reactor safety studies, this information is needed to assess the potential for downward fuel relocation through narrow passageways.

To investigate the transient behavior of the heat-generating substance, the equations governing the solid–liquid interface motion and the temperature of the system are transformed in terms of several dimensionless variables. In the transformed coordinates, the location of the solid–liquid interface is immobilized. The transformed equations are then solved numerically to compute the rates of growth and decay as well as the lifetime of the frozen layer (or 'crust') as functions of the rate of heat generation in the slab and the thermal properties of the wall and the heat-generating substance. For the case in which melting occurs before the entire slab freezes solid, the maximum crust thickness is also determined.

2. PROBLEM FORMULATION

Consider a hot liquid with uniformly distributed volumetric heat generation rate q, confined between two semi-infinite cold walls. Initially, the liquid is at a uniform temperature, T_0, equal to or above its freezing point T_m and the two walls are at the same temperature T_∞. The value of T_∞ is assumed to be much less than the value of T_m so that freezing of the liquid immediately proceeds on both walls.† For convenience, both walls are assumed to be made up of the same materials so that their thermal properties are identical. With respect to

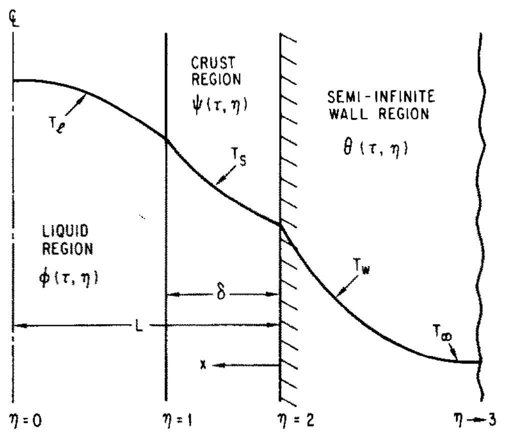

FIG. 1. Schematic of the liquid–crust–wall system, indicating coordinate system used.

the centerline of the slab, the heat transfer process is symmetrical. Thus only half of the system needs to be investigated. As shown in Fig. 1, a solidification front at $x = \delta(\geqslant 0)$ will propagate toward the center of the liquid from the wall as long as the latent heat of fusion and the heat generated in the slab can be effectively conducted into the wall. As the thermal boundary layer penetrates deeper and deeper into the wall, however, the conductive wall heat flux may become less than the rate of heat generation in the slab. Solidification ceases to proceed beyond this time and, in fact, re-melting of the solidified layer will occur, causing the crust to shrink and eventually disappear. To model this growth and decay phenomena, the following simplifying assumptions are used:

(1) The rate of heat generation is uniformly distributed throughout the liquid and the crust regions and remains constant over the lifetime of the crust.

(2) All physical properties are considered to be independent of temperature. To minimize the number of parameters, the same properties are assumed for the liquid and the crust. In particular, there is no volume change upon freezing or melting.

(3) The solidification or melting front is sharp and planar on the scale of the crust thickness and the temperature at the liquid–solid interface is equal to the freezing temperature. The effect of supercooling [11] will not be considered here.

(4) Over the lifetime of the crust, there is no boiling of the liquid nor melting of the wall.‡

(5) The width of the slab and the temperature drop across the liquid region are such that no free convection motion is induced in the liquid. Furthermore, the effect of radiation is negligible or can be accounted for by using the concept of effective thermal conductivity.

With the above assumptions, the equations governing the temperatures of the liquid, the solid crust,

* This is true even if the heat-generating substance is initially solid. In that case there is no freezing occurring in the system. The slab will be subcooled below its melting point in the early stage. However, reheating of the slab will occur later on due to decay heat generation, and eventually melting will proceed. This situation will be discussed in the text.

† In nuclear reactor safety studies, for example, T_∞ has a value ranging from 500 to 1000 K whereas T_m is of the order of 3000 K.

‡ This requires that the centerline temperature of the slab and the surface temperature of the wall be less than the liquid boiling point and the wall melting point, respectively.

and the wall, i.e. T_l, T_s, and T_w, respectively, as well as the liquid–solid interface location, $\delta(t)$, can be written as follows:

$$\frac{\partial T_l}{\partial t} = \alpha_s \frac{\partial^2 T_l}{\partial x^2} + \frac{q}{\rho_s C p_s}, \quad \delta \leqslant x \leqslant L, \tag{1}$$

$$\frac{\partial T_s}{\partial t} = \alpha_s \frac{\partial^2 T_s}{\partial x^2} + \frac{q}{\rho_s C p_s}, \quad 0 \leqslant x \leqslant \delta, \tag{2}$$

$$\frac{\partial T_w}{\partial t} = \alpha_w \frac{\partial^2 T_w}{\partial x^2}, \quad x \leqslant 0. \tag{3}$$

$$t = 0: \quad T_l = T_0, \quad T_w = T_\infty, \quad \delta = 0 \quad (T_0 > T_m > T_\infty), \tag{4a}$$

$$x = L: \quad \frac{\partial T_l}{\partial x} = 0, \tag{4b}$$

$$x = \delta: \quad T_l = T_s = T_m, \tag{4c}$$

$$\frac{d\delta}{dt} = \frac{k_s}{\rho_s \Delta H_f}\left(\frac{\partial T_s}{\partial x} - \frac{\partial T_l}{\partial x}\right), \tag{4d}$$

$$x = 0: \quad T_s = T_w, \quad k_s \frac{\partial T_s}{\partial x} = k_w \frac{\partial T_w}{\partial x}, \tag{4e}$$

$$x \to -\infty: \quad T_w = T_\infty. \tag{4f}$$

In the above formulation, the spatial coordinate x is measured from the wall toward the liquid region. The same properties have been assumed for the liquid and the crust. The boundary condition given by equation (4b) represents the symmetric nature of the problem. The value of L is finite and q is regarded as a constant, independent of space and time.

The above system of equations is valid as long as $\delta < L$. This condition applies to the situation in which melting of the crust occurs before the entire region freezes solid. For the case in which q is small and k_w is large, the solidification front may reach the center of the slab before melting commences. In this case, the liquid region will disappear altogether for a period of time. Meanwhile, the temperature at the center of the slab will fall below the melting point of the crust, i.e. $T_s(x = \delta = L) < T_m$, and there is no latent heat absorbed or released in the system. During this 'subcooling' period in which $\delta = L$ and $T_s(x = L) < T_m$, numerical solution must be continued with the following set of governing equations:

$$\frac{\partial T_s}{\partial t} = \alpha_s \frac{\partial^2 T_s}{\partial x^2} + \frac{q}{\rho_s C p_s}, \quad 0 \leqslant x \leqslant L, \tag{5}$$

$$\frac{\partial T_w}{\partial t} = \alpha_w \frac{\partial^2 T_w}{\partial x^2}, \quad x \leqslant 0, \tag{6}$$

$$x = L: \quad \frac{\partial T_s}{\partial x} = 0, \tag{7a}$$

$$x = 0: \quad T_s = T_w, \quad k_s \frac{\partial T_s}{\partial x} = k_w \frac{\partial T_w}{\partial x}, \tag{7b}$$

$$x \to -\infty: \quad T_w = T_\infty. \tag{7c}$$

The temperature distribution of the system at the end of the freezing period, i.e. at the time when $\delta \to L$, will serve as the initial condition for the above system.

For a non-zero value of q, reheating of the frozen slab will occur in time as the wall heat flux becomes small enough. Ultimately, the centerline temperature will rise back up to the value of T_m and melting of the crust will commence. Equations (1)–(4) may be used to describe the melting behavior, with the initial condition, equation (4a), being replaced by the system condition given by the solution of equations (5)–(7) at the end of the reheating period. It is quite obvious that over the lifetime of the crust, the system will go through three transitions. The first transition is from freezing to subcooling, the second from subcooling to reheating, and the third from reheating to melting.* These transitions occur because there is heat generation in the system and the heat flux conducted through the crust into the wall is diminishing in time.

3. METHOD OF SOLUTION

We have shown that in various periods of time the transient behavior of the system must be described by different sets of governing equations. This feature and the non-linear character associated with the phase change process make the present problem extremely complex. It is not feasible to seek approximate analytical solutions. For this reason, the governing equations are solved numerically. To facilitate the task of computation and to identify the various controlling parameters, the system is transformed in terms of the following dimensionless variables:

$$\tau = \frac{\alpha_s t}{L^2}, \quad \Delta = \frac{\delta}{L} \quad \text{for} \quad 0 \leqslant \Delta(\tau) \leqslant 1, \tag{8}$$

$$\phi = \frac{T_l - T_\infty}{T_m - T_\infty}, \quad \psi = \frac{T_s - T_\infty}{T_m - T_\infty}, \quad \theta = \frac{T_w - T_\infty}{T_m - T_\infty}, \tag{9}$$

$$\eta = \frac{1 - x/L}{1 - \Delta} \quad \text{for} \quad \Delta \leqslant \frac{x}{L} \leqslant 1, \tag{10a}$$

$$\eta = 2 - \frac{1}{\Delta}\frac{x}{L} \quad \text{for} \quad 0 \leqslant \frac{x}{L} \leqslant \Delta, \tag{10b}$$

$$\eta = 3 - \exp\left(\frac{\beta x}{L}\right) \quad \text{for} \quad \frac{x}{L} \leqslant 0, \tag{10c}$$

where β, an input stretching parameter, is a positive constant. In the dimensionless space, the liquid–solid interface remains fixed at $\eta = 1$. Moreover, the liquid region is contained between $0 \leqslant \eta \leqslant 1$, the crust region between $1 \leqslant \eta \leqslant 2$, and the wall region between $2 \leqslant \eta \leqslant 3$. The dimensionless temperatures ϕ, ψ, and θ for the three separate regions, respectively, are now dependents of τ and η.

In dimensionless variables, equations (1) and (4)

* Should the heat-generating substance be initially solid, there would be only two transitions in the system, namely from subcooling to reheating and from reheating to melting.

become

$$\frac{\partial \phi}{\partial \tau} = -\eta \frac{1}{1-\Delta}\frac{d\Delta}{d\tau} + \frac{1}{(1-\Delta)^2}\frac{\partial^2 \phi}{\partial \eta^2} + A, \quad 0 \leqslant \eta \leqslant 1, \tag{11}$$

$$\frac{\partial \psi}{\partial \tau} = -(2-\eta)\frac{1}{\Delta}\frac{d\Delta}{d\tau}\frac{\partial \psi}{\partial \eta} + \frac{1}{\Delta^2}\frac{\partial^2 \psi}{\partial \eta^2} + A, \quad 1 \leqslant \eta \leqslant 2, \tag{12}$$

$$\frac{\partial \theta}{\partial \tau} = \beta^2(3-\eta)\left(\frac{\alpha_w}{\alpha_s}\right)\left[-\frac{\partial \theta}{\partial \eta} + (3-\eta)\frac{\partial^2 \theta}{\partial \eta^2}\right], \quad 1 \leqslant \eta \leqslant 3, \tag{13}$$

$$\tau = 0: \quad \phi = \phi_0, \quad \theta = 0, \quad \Delta = 0, \tag{14a}$$

$$\eta = 0: \quad \frac{\partial \phi}{\partial \eta} = 0, \tag{14b}$$

$$\eta = 1: \quad \phi = \psi = 1, \tag{14c}$$

$$\frac{d\Delta}{d\tau} = Ste\left[\frac{1}{1-\Delta}\frac{\partial \phi}{\partial \eta} - \frac{1}{\Delta}\frac{\partial \psi}{\partial \eta}\right], \tag{14d}$$

$$\eta = 2: \quad \psi = \theta, \quad \frac{\partial \psi}{\partial \eta} = \beta\left(\frac{k_w}{k_s}\right)\Delta\frac{\partial \theta}{\partial \eta}, \tag{14e}$$

$$\eta = 3: \quad \theta = 0, \tag{14f}$$

where A, ϕ_0, and Ste are the dimensionless heat-generation rate, the liquid superheat factor, and the Stefan number defined respectively by

$$A = \frac{qL^2}{k_s(T_m - T_\infty)}, \tag{15a}$$

$$\phi_0 = \frac{T_0 - T_\infty}{T_m - T_\infty} (\geqslant 1), \tag{15b}$$

and

$$Ste = \frac{Cp_s(T_m - T_\infty)}{\Delta H_f}. \tag{15c}$$

The above system of equations applies to the case of freezing and melting for which $0 \leqslant \Delta \leqslant 1$. When the entire slab freezes solid, i.e. when the liquid region disappears, we have $\Delta = 1$, $d\Delta/d\tau = 0$, and $\psi(\tau, 1) \leqslant 1$. Numerical computation must be continued with equations (5)–(7) which, after transformation become

$$\frac{\partial \psi}{\partial \tau} = \frac{\partial^2 \psi}{\partial \eta^2} + A, \quad 1 \leqslant \eta \leqslant 2 \tag{16}$$

$$\frac{\partial \theta}{\partial \tau} = \beta^2(3-\eta)\left(\frac{\alpha_w}{\alpha_s}\right)\left[-\frac{\partial \theta}{\partial \eta} + (3-\eta)\frac{\partial^2 \theta}{\partial \eta^2}\right], \quad 2 \leqslant \eta \leqslant 3, \tag{17}$$

$$\eta = 1: \quad \frac{\partial \psi}{\partial \eta} = 0, \tag{18a}$$

$$\eta = 2: \quad \psi = \theta, \quad \frac{\partial \psi}{\partial \eta} = \beta\left(\frac{k_w}{k_s}\right)\frac{\partial \theta}{\partial \eta}, \tag{18b}$$

$$\eta = 3: \quad \theta = 0. \tag{18c}$$

Transition from complete freezing to subcooling occurs when $\Delta \to 1$ whereas transition from reheating to melting occurs when $\psi(\tau, 1) = 1$ with $\partial\psi/\partial\tau > 0$. Inspection of equations (11)–(18) indicates that $\Delta(\tau)$ is controlled by five parameters, namely, A, ϕ_0, Ste, k_w/k_s, and α_w/α_s.

At sufficiently small times, we have $(1-\Delta)/\Delta \to \infty$. The liquid region may be regarded as semi-infinite. Also, the effect of heat generation may be neglected since $A\tau \to 0$. The following similarity solutions are valid:

$$\Delta = 2\lambda\sqrt{\tau}, \tag{19}$$

$$\phi = \phi_0 - \frac{\phi_0 - 1}{\operatorname{erfc}\lambda}\operatorname{erfc}\left[\frac{1-(1-2\lambda\sqrt{\tau})\eta}{2\sqrt{\tau}}\right], \quad 0 \leqslant \eta \leqslant 1, \tag{20}$$

$$\psi = \frac{1+(k_w/k_s)(\alpha_s/\alpha_w)^{1/2}\operatorname{erf}[(2-\eta)\lambda]}{1+(k_w/k_s)(\alpha_s/\alpha_w)^{1/2}\operatorname{erf}\lambda}, \quad 1 \leqslant \eta \leqslant 2, \tag{21}$$

$$\theta = \frac{1-\operatorname{erf}\{(1/2\beta\sqrt{\tau})\ln[1/(3-\eta)]\}}{1+(k_w/k_s)(\alpha_s/\alpha_w)^{1/2}\operatorname{erf}\lambda}, \quad 2 \leqslant \eta \leqslant 3, \tag{22}$$

where the solidification constant λ is given implicitly by

$$\frac{e^{-\lambda^2}}{(k_s/k_w)(\alpha_w/\alpha_s)^{1/2}\operatorname{erf}\lambda} - \frac{(\phi_0-1)\,e^{-\lambda^2}}{\operatorname{erfc}\lambda} = \frac{\sqrt{(\pi\lambda)}}{Ste}. \tag{23}$$

Equations (19)–(23), which are exact expressions for ϕ, ψ, and θ for $\tau \to 0$, provide the initial starting temperature distribution of the system for numerical computation.

The dimensionless governing equations are solved by using the method of collocation with Hermite splines as approximating functions and Gaussian quadrature points as the collocation points. The details of the method can be found in ref. [12]. The method has been used for solution of a wide variety of problems in heat and mass transfer with and without phase change and is known to produce highly accurate results. An optimum value of $\beta = 0.5$ is used in the present study.

4. RESULTS AND DISCUSSION

We have identified five parameters which control the freezing and melting behavior of the system. These are the heat generation rate, A, the liquid superheat factor, ϕ_0, the Stefan number, Ste, and the thermal property ratios, k_w/k_s and α_w/α_s. Obviously, it is not possible to present the combined effect of all of these parameters in a simple manner. For this reason, a parametric study of the problem is performed and the individual effect of each parameter is investigated separately. To simplify the presentation, we further restrict ourselves to the case of $\alpha_w/\alpha_s = k_w/k_s$.*

* Physically, both parameters, α_w/α_s and k_w/k_s, signify the importance of the wall effect. Previous studies [13, 14] of this class of problems indicate that the two parameters can actually be combined into one. Thus, there is no loss of generality by using this assumption.

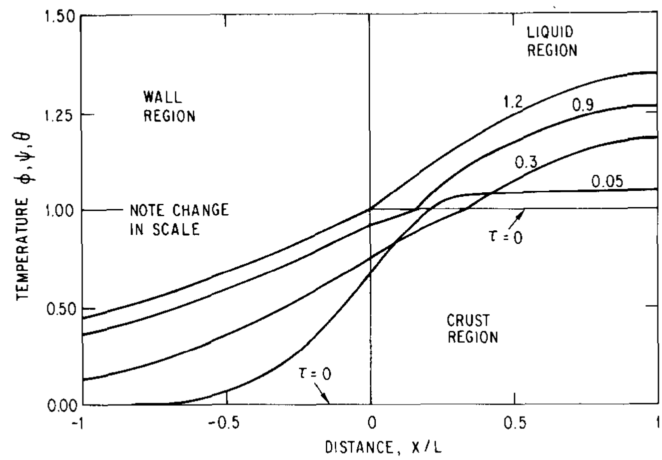

FIG. 2. Transient temperature distribution of the system for the case of $A = 1, \phi_0 = 1, Ste = 2$, and $k_w/k_s = 1$.

The transient temperature distribution of the system for the case of $A = 1$, $\phi_0 = 1$, $Ste = 2$, and $k_w/k_s = 1$ is shown in Fig. 2. To show how the solidification front and the melting front proceed in time, the dimensionless temperatures are plotted against the physical distance from the surface of the wall normalized by the half width of the slab. At $\tau = 0$, the wall temperature θ is equal to zero and the liquid temperature ϕ is equal to unity. Since the wall heat flux is very large at small times, freezing of the liquid proceeds relatively rapidly. At $\tau = 0.05$, the solidification front has already reached the location of $x/L = 0.23$. In this early stage of crust growth, the thermal boundary layer has not yet penetrated through the entire liquid region. Consequently, the centerline temperature rises adiabatically according to $(1+\tau)$. The thickness of the thermal boundary layer in the wall is only $0.65L$. At $\tau = 0.3$, the crust achieves its maximum thickness of $0.34L$. At this moment, there is no latent heat released or absorbed in the system. As the thermal boundary layer in the wall grows thicker, the heat generated in the slab can no longer be removed effectively into the wall. As a result, melting of the crust commences. Meanwhile, the rate of increase of the centerline temperature drops below unity. With the wall heat flux diminishing in time, melting of the crust is accelerating. This can be seen from the position of the temperature profile at $\tau = 0.9$. Melting is complete at $\tau = 1.2$ and the lifetime of the crust is over. It should be noted that during the growth period, $0 \leqslant \tau \leqslant 0.3$, the temperature gradient of the crust at the moving interface is always larger than the temperature gradient of the liquid.* The reverse is true during the period of decay, $0.3 \leqslant \tau \leqslant 1.2$.

The dependence of the transient crust thickness on the dimensionless heat generation rate is displayed in Fig. 3 for the case of $\phi_0 = 1$, $Ste = 1$, and $k_w/k_s = 1$. It is found that the growth and decay behavior of the crust is highly sensitive to the level of internal heating. As expected, we may substantially prolong the crust lifetime and increase the maximum crust thickness† by decreasing the value of A. At the limit of $A = 0$, there is no melting of the crust. The slab remains frozen indefinitely. For the general case of $A > 0$, the situation is quite different. No matter how small the value of A is, the crust will always decay after fully grown and ultimately disappear due to melting, although the crust lifetime may be extremely large. Actually, if $0 \leqslant A \leqslant 0.1$, a period of subcooling without phase change will follow after the entire region is frozen and a period of reheating without phase change will precede before melting of the slab commences. The same situation may arise in the case in which the heat-generating substance is initially solid. However, if $A > 0.1$, melting of the crust will begin immediately after the crust thickness peaks, as can be seen in Fig. 3. It should be noted that the transition from freezing to melting is very slow when $0.1 \leqslant A \leqslant 0.3$. Under these circumstances, computation‡ has to be terminated artificially before the end of the crust lifetime.

Figure 4 presents the effects of liquid superheat, thermal conductivity ratio, and the Stefan number on the transient crust thickness with the internal heat-generation rate fixed at $A = 1$. Case (i) corresponds to the conditions: $\phi_0 = 1$, $Ste = 1$, and $k_w/k_s = 1$; case (ii) to: $\phi_0 = 1$, $Ste = 2$, and $k_w/k_s = 1$; case (iii) to: $\phi_0 = 1$, $Ste = 1$, and $k_w/k_s = 5$; and case (iv) to: $\phi_0 = 2$, $Ste = 1$, and $k_w/k_s = 5$. These cases are chosen such that melting of the crust occurs before the entire slab freezes solid.§ Comparison between case (i) and case (ii) indicates that an increase in the value of Ste leads to an

* In Fig. 2 the temperature scale for the crust region is twice as large as the scale for the liquid region.

† The peak value of this is, of course, limited by the half width of the slab, namely, $\Delta_m \leqslant 1$.

‡ Since the process of crust decay is accelerating toward the end of the melting period, it is not feasible to simply increase the magnitude of the time step in the numerical computation.

§ This choice enables us to clearly show the effects of ϕ_0, Ste, and k_w/k_s on the maximum crust thickness as well as the time at which the crust thickness peaks.

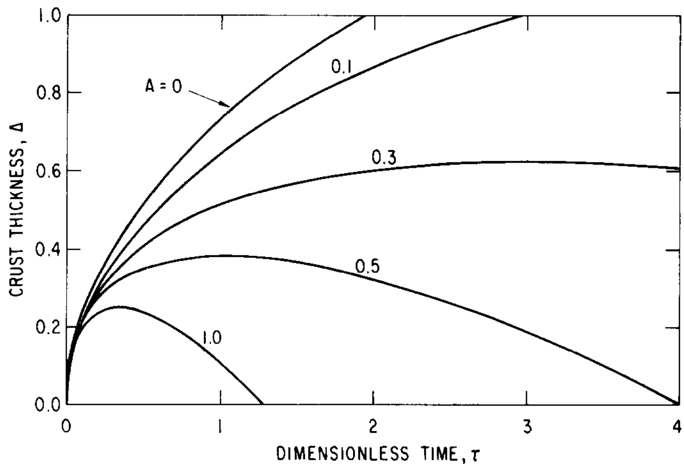

FIG. 3. Effect of internal heat generation on the growth and decay of the crust.

increase in the crust thickness. However, both the time at which the crust thickness peaks and the total crust lifetime are about the same in both cases. This behavior is caused by the fact that the time scales involved in the growth and decay process are determined mainly by the rate of internal heating of the slab and external cooling due to the wall. The latent heat of fusion released or absorbed in the system is simply a result of the imbalance between the rates of heating and cooling. Since the heating rate depends only on the value of A and the cooling rate depends principally on the value of k_w/k_s, the time scales are insensitive to the value of Ste. On the other hand, there is more crust deposited or melted in a given period of time as the latent heat effect becomes weaker. Comparison between case (i) and case (iii) indicates that an increase in the value of k_w/k_s leads to an increase in the crust thickness as well as to a much longer crust lifetime. Obviously, solidification would proceed at a faster rate and for a longer time whereas melting would proceed at a slower rate but for a longer time as the wall cooling effect becomes stronger. Finally, comparison between case (iii) and case (iv) indicates that an increase in the value of ϕ_0 leads to a decrease in the crust thickness as well as to a slightly shorter crust lifetime. The time at which the crust thickness peaks, however, is almost unchanged. It appears that the transition from freezing to melting depends mainly* on the value of k_w/k_s in the cases studied here.

The effect of internal heat generation on the maximum crust thickness and the crust lifetime is further shown in Fig. 5. In this figure, the liquid superheat factor is fixed at $\phi_0 = 1$ while the conductivity ratio at $k_w/k_s = 1$. As expected, both the maximum crust thickness, Δ_m, and the crust lifetime, τ_1, are monotonically decreasing functions of the heat-generation rate. The values of Δ_m and τ_1 are found to be quite sensitive to the variation in A as the heat generation drops below the level of $A = 1$. Physically, the product $A\tau_1$ represents the total heat generated in the system over the lifetime of the crust. For $A < 1$, we have $\tau_1 > 1$ so that the effect of heat generation is

* Note that the value of A is fixed at unity in all four cases. In general, the transition would depend on both A and k_w/k_s.

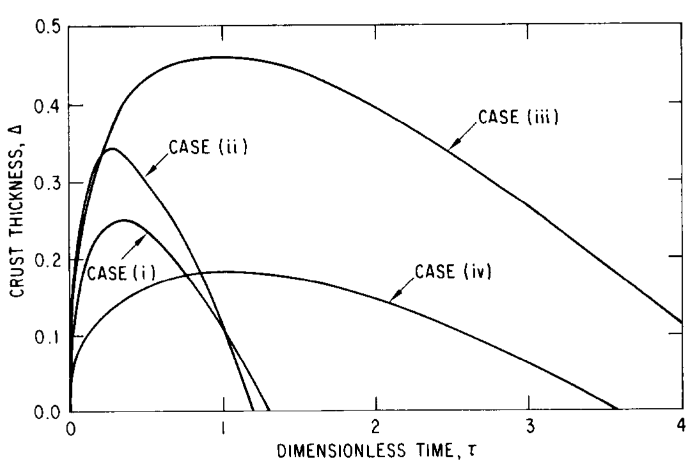

FIG. 4. Growth and decay of the crust under various system conditions.

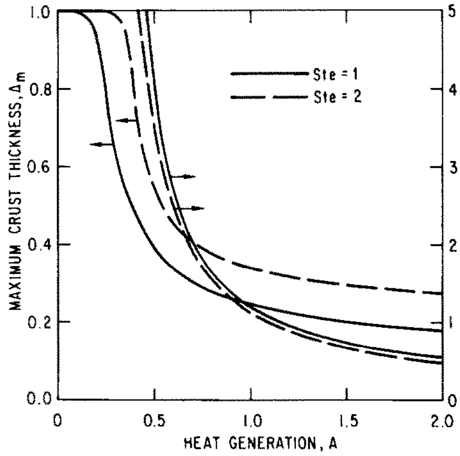

FIG. 5. Effect of internal heat generation on the maximum crust thickness and the total crust lifetime with $\phi_0 = 1$, $k_w/k_s = 1$, and the Stefan number as a parameter.

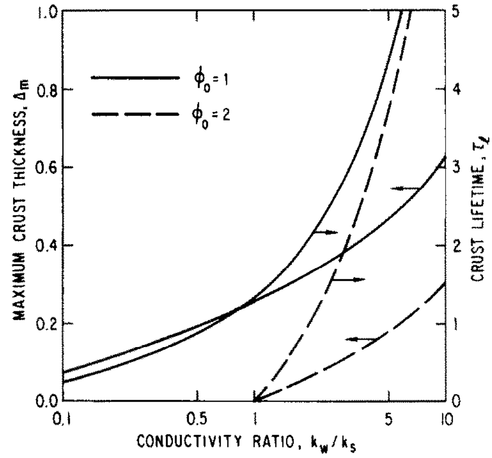

FIG. 6. Effect of the wall-to-crust thermal conductivity ratio on the maximum crust thickness and the total crust lifetime with $A = 1$, $Ste = 1$, and the liquid superheat factor as a parameter.

amplified. Since the wall cooling effect is diminishing* in time, the growth and decay process is strongly affected by the value of A. On the other hand, for $A > 1$, we have $T_l < 1$ so that the effect of heat generation is reduced. Thus, Δ_m and T_l are less sensitive to the change in the value of A. It should be noted that for A less than a certain value, the entire slab would freeze solid before melting of the crust occurs. For the case of $Ste = 1$ (solid line) this value is 0.1 whereas for the case of $Ste = 2$ (dashed line) this value is 0.3.

Figure 6 displays the effect of wall interaction on the maximum crust thickness and the crust lifetime with the rate of heat generation fixed at $A = 1$ and the Stefan number at $Ste = 1$. In contrast to the case shown in Fig. 5, here both Δ_m and τ_l are monotonically increasing functions of k_w/k_s. Should the wall be a perfect conductor, i.e. $k_w/k_s \to \infty, \tau_l$ would approach infinity and Δ_m would be equal to unity.† On the other hand, both τ_l and Δ_m would be identically zero should the wall be a poor conductor. Under these circumstances, there would be no phase change occurring in the slab. Physically this corresponds to the situation in which the contact temperature between the wall and the liquid is equal to or above the solidification point of the liquid, namely

$$\frac{k_s\alpha_s^{-1/2}\phi_0}{k_s\alpha_s^{-1/2}+k_w\alpha_w^{-1/2}} \geqslant 1 \quad \text{or} \quad \phi_0 \geqslant \left[1+\frac{k_w}{k_s}\left(\frac{\alpha_w}{\alpha_s}\right)^{-1/2}\right]. \tag{24}$$

Assuming $k_s = \alpha_s$ and $k_w = \alpha_w$, we have

$$\frac{k_w}{k_s} \leqslant (\phi_0-1)^2. \tag{25}$$

When this condition is met, there will be no crust formation in the system. For the case of $\phi_0 = 2$ (dashed line), this situation arises as $k_w/k_s \leqslant 1$ whereas for the case of $\phi_0 = 1$ (solid line), the criterion is $k_w/k_s = 0$.‡

5. EFFECT OF SYSTEM GEOMETRY

Thus far, we have restricted ourselves to the case of a finite slab. We have shown that a permanently frozen (coolable) system could be obtained only in two limiting cases, i.e. those of $A = 0$ and $(k_w/k_s) \to \infty$. More generally, the crust would decay and finally disappear due to internal heating of the slab. This would eventually lead to melting of the wall. The situation, however, is quite different if the heat-generating substance has a spherical geometry.§ It is known that, in a spherical system, the wall heat flux approaches a finite value at very large times. If the rate of heat generation in the substance is less than this value, a steady state may be achieved in the system with the heat-generating substance being frozen indefinitely. At steady state, the wall temperature, θ, and the temperature of the frozen substance, ψ, are governed respectively by

$$\frac{d^2\theta}{dR^2}+\frac{2}{R}\frac{d\theta}{dR} = 0, \quad R \geqslant 1, \tag{26}$$

$$\frac{d^2\psi}{dR^2}+\frac{2}{R}\frac{d\psi}{dR}+A = 0, \quad 0 \leqslant R \leqslant 1, \tag{27}$$

$$R = 0; \quad d\psi/dR = 0, \tag{28a}$$

$$R = 1: \quad \theta = \psi; \quad k_w\, d\theta/dR = k_s\, d\psi/dR, \tag{28b}$$

$$R \to \infty: \quad \theta = 0, \tag{28c}$$

where $R = r/L$ is the dimensionless radial coordinate measured from the center of the sphere. The solutions

* At large τ, the wall heat flux decreases very slowly with time. It follows that a small decrease in A would result in a large increase in τ_l (see Fig. 5).

† Thus the entire slab would be frozen indefinitely.

‡ This corresponds to the case in which the wall is a perfect insulator.

§ This situation arises, for example, in the treatment (such as underground burial) of radioactive waste materials.

are

$$\theta = \frac{A}{3}\left(\frac{k_w}{k_s}\right)^{-1}\frac{1}{R}, \quad R \geqslant 1, \tag{29}$$

$$\psi = \frac{A}{6}\left[1+2\left(\frac{k_w}{k_s}\right)^{-1} - R^2\right], \quad 0 \leqslant R \leqslant 1. \tag{30}$$

For the heat-generating sphere to remain frozen at all times, its maximum temperature, which occurs at its center, must be equal to or less than the melting point of the substance. Mathematically, this requires $\psi(0) \leqslant 1$ or from equation (30)

$$A \leqslant \frac{6}{1+2(k_w/k_s)^{-1}}. \tag{31}$$

As long as A is smaller than this value, a permanently frozen system can be obtained. Note that this critical value is a function of the wall-to-crust conductivity ratio alone. For the case of $k_w/k_s = 1$, for example, the sphere, once frozen, would remain solid provided that $A \leqslant 2$. Referring to Fig. 3, it is obvious that a heat-generating sphere can be cooled more easily than a heat-generating slab. Physically, this result is quite expected. The available heat transfer surface area per unit volume of the heat-generating substance is much larger for the case of a heat-generating sphere than for the case of a heat-generating slab.*

6. SUMMARY AND CONCLUSIONS

We have investigated numerically the transient behavior of a heat-generating slab confined between two semi-infinite cold walls. We have shown that a permanently frozen system cannot be obtained as long as the heat-generation rate is not identically zero. Owing to the fact that the slab is heated internally at a constant rate whereas the heat flux conducted through the crust into the wall is diminishing in time, the solidified layer, once formed, must decay and ultimately disappear via melting. Depending on the magnitude of various controlling parameters, melting of the crust may occur before the slab freezes solid or it may occur after the entire region is frozen. In the former case, a transition from freezing directly to melting will take place in the system. In the latter case, however, a period of subcooling and reheating of the slab without phase change will prevail in the system before the onset of melting. The system will go through three transitions over the lifetime of the crust, i.e. from freezing to subcooling, from subcooling to reheating, and from reheating to melting. It follows that the transient behavior of the system must be described by different sets of governing equations in various stages of the crust life. Numerical solutions to these equations have been obtained and the effects of various controlling parameters have been identified. Although the maximum crust thickness is a function of the Stefan number and the liquid superheat factor, the time at which the crust thickness peaks and the crust lifetime are found to depend principally on the crust-to-wall conductivity ratio and the rate of heat generation of the slab. With respect to the case of a heat-generating sphere, a heat-generating slab is shown to be much more difficult to cool.

* With respect to the wall which is infinite in extent, a heat-generating sphere is equivalent to a point source whereas a heat-generating slab is equivalent to a plane source. It is of course easier to conduct heat away from a point.

Acknowledgements—The authors wish to thank Ms. Alice Townsend for typing the manuscript. The work was performed under the auspices of the U.S. Department of Energy.

REFERENCES

1. J. Stefan, Uber die theorie der eisbildung insbecondere uber die eisbildung in polarmaere, *Ann. Phys. Chem.* **43**, 269–286 (1891).
2. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (2nd edn.). Clarendon Press, London (1959).
3. S. G. Bankoff, Heat conduction or diffusion with change of phase, *Adv. Chem. Engng* **5**, 75–150 (1964).
4. J. C. Muehlbauer and J. E. Sunderland, Heat conduction with freezing or melting, *Appl. Mech. Rev.* **18**, 951–959 (1965).
5. D. G. Wilson, A. D. Solomon and P. T. Boggs, *Moving Boundary Problems*. Academic Press, New York (1978).
6. M. N. Özişik, *Heat Conduction*. Wiley, New York (1980).
7. V. J. Lunardini, *Heat Transfer in Cold Climates*. Van Nostrand Reinhold, New York (1981).
8. A. L. Ruoff, An alternative solution to Stefan's problem, *Q. Appl. Math.* **16**, 197–201 (1958).
9. T. R. Goodman and J. Shea, The melting of finite slabs, *J. Appl. Mech.* **27**, 16–24 (1960).
10. S. H. Cho and J. E. Sunderland, Heat conduction problems with melting or freezing, *J. Heat Transfer* **91**, 421–426 (1969).
11. C. A. Knight, *The Freezing of Supercooled Liquids*. Van Nostrand, Princeton, New Jersey (1967).
12. T. C. Chawla, D. R. Pedersen, G. Leaf, W. J. Minkowycz and A. R. Shoumen, Adaptive collocation method for simultaneous heat and mass diffusion with phase change, *J. Heat Transfer* (to be published).
13. M. Epstein, The growth and decay of a frozen layer in forced flow, *Int. J. Heat Mass Transfer* **19**, 1281–1287 (1976).
14. F. B. Cheung, Transient non-similarity non-linear heat diffusion solutions, *J. Heat Transfer* **105**, 295–301 (1983).

LES EFFETS DE CREATION DE CHALEUR ET D'INTERACTION PARIETALE SUR LE GEL ET LA FUSION DANS UNE PLAQUE

Résumé—Les mécanismes de gel et de fusion dans une plaque avec source de chaleur et limitée par deux parois froides semi-infinies sont étudiés numériquement. On emploie la méthode de collocation pour résoudre les différents systèmes d'équations qui décrivent le comportement variable du système pendant différentes périodes de temps. Suivant l'intensité de la création de chaleur interne et les propriétés thermiques des parois et de la plaque, différents changements apparaissent dans le système. Ces changements, comme l'indique la position variable de l'interface solide–liquide, incluent des transitions du gel à la fusion directe, du gel au refroidissement sans changement de phase, du refroidissement au chauffage sans changement de phase, et du chauffage à la fusion. Des prédictions numériques sur l'apparition de ces transitions, sur les vitesses de congélation et de fusion et sur la durée des phénomènes transitoires sont obtenues en fonction des plusieurs paramètres adimensionnels. Une comparaison est faite dans le cas d'une sphère afin d'explorer l'effet de la géométrie du système.

DER EINFLUSS VON WÄRMEERZEUGUNG UND WANDEINFLUSS AUF ERSTARREN UND SCHMELZEN IN EINER ENDLICHEN PLATTE

Zusammenfassung—Die Erstarrungs- und Schmelzvorgänge wie sie in einer wärmeerzeugenden Platte ablaufen, die von zwei halbunendlichen kalten Wänden begrenzt wird, werden numerische untersucht. Zur Lösung der verschiedenen Gleichungssysteme, die das instationäre Verhalten des Systems zu verschiedenen Zeiten beschreiben, wird die Kollokationsmethode angewandt. Im System können verschiedene Vorgänge ablaufen, je nach der internen Wärmeerzeugung und den thermischen Eigenschaften von Platte und Begrenzungswänden. Wie bereits die Bewegung der Phasengrenze erkennen läßt, umfassen die Vorgänge den direkten Übergang vom Erstarren zum Schmelzen, vom Erstarren zum Abkühlen ohne Phasenwechsel, vom Abkühlen zum Aufheizen ohne Phasenwechsel und vom Aufheizen zum Schmelzen. Es werden numerische Berechnungen des Auftretens dieser Übergänge dargestellt; die Schmelz- und Erstarrungsgeschwindigkeit und die Dauer der Übergänge erhält man als Funktion von verschiedenen bestimmenden dimensionslosen Parametern des Systems. Um weiter den Einfluß der Systemgeometrie zu untersuchen, werden die Ergebnisse mit denen einer wärmeerzeugenden Kugel verglichen.

ВЛИЯНИЕ ТЕПЛОВЫДЕЛЕНИЯ И ВЗАИМОДЕЙСТВИЯ СО СТЕНКОЙ НА ПРОЦЕССЫ ЗАТВЕРДЕВАНИЯ И ПЛАВЛЕНИЯ В ПЛИТЕ КОНЕЧНЫХ РАЗМЕРОВ

Аннотация—Проведено численное исследование процессов затвердевания и плавления в тепловыделяющей плите, ограниченной двумя полубесконечными холодными стенками. Для решения различных исходных уравнений, описывающих нестационарное поведение системы в разные моменты времени, используется метод коллокаций. В зависимости от интенсивности внутреннего тепловыделения и тепловых характеристик стенки и плиты в системе может происходить ряд изменений, которые, как показывает изменяющееся положение границы раздела твердое тело жидкость, включают переходы от затвердевания непосредственно к плавлению, от затвердевания к охлаждению без изменения фаз, от охлаждения к нагреванию без изменения фаз и от нагревания к плавлению. Численно исследовано наличие таких переходов, а также определены скорость затвердевания и плавления и длительность переходных процессов как функции нескольких контрольных безразмерных параметров системы. Для определения влияния геометрии системы проведено сравнение со случаем тепловыделяющей сферы.